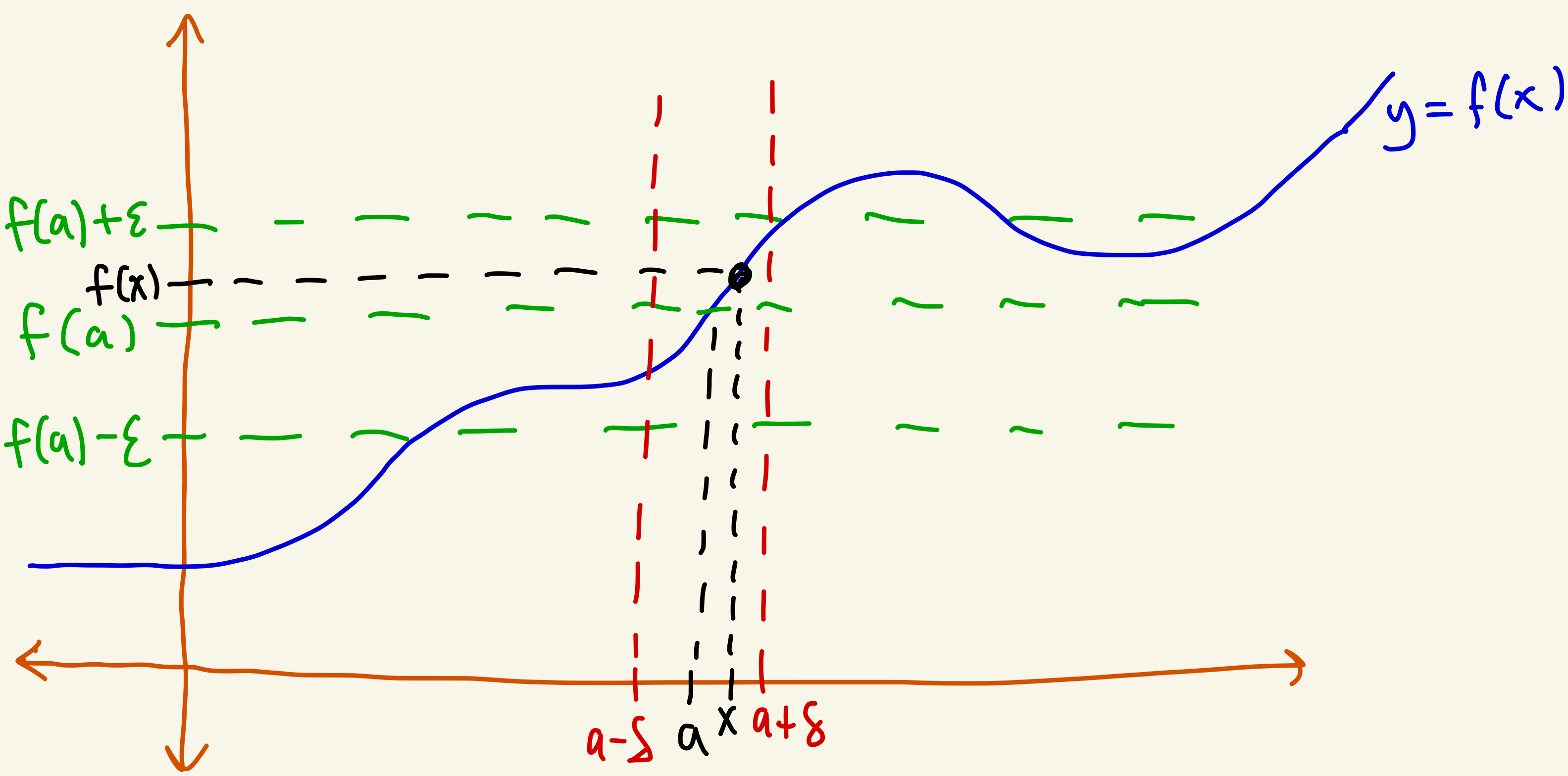
Math 4650

Topic 5 – Continuity

Def: Let $D \subseteq \mathbb{R}$, $a \in \mathbb{R}$, and $f: D \to \mathbb{R}$. We say that f is <u>continuous at a</u> if for every $\varepsilon > 0$ there exists $\delta > 0$ so that if $x \in D$ and $|x-a| < \delta$ then $|f(x) - f(a)| < \varepsilon$.



Note: a is in the domain of f. And x is allowed to equal a.

If $A \subseteq D$ and f is continuous at all $a \in A$, then we say that f is <u>continuous on A</u>.

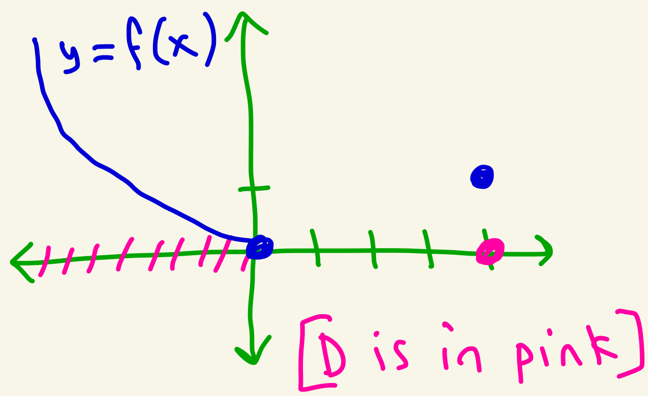
To get an idea of what the above means let's have an example in mind as we go through the two scenarios.

Keep this function in mind for the note below.

$$D = (-\infty, 0] \cup \{4\}$$

$$f : D \to \mathbb{R}$$

$$f(x) = \begin{cases} x^2 & \text{if } x \leq 0 \\ 1 & \text{if } x = 4 \end{cases}$$



Note: There are two cases for the definition of continuity of f at a.

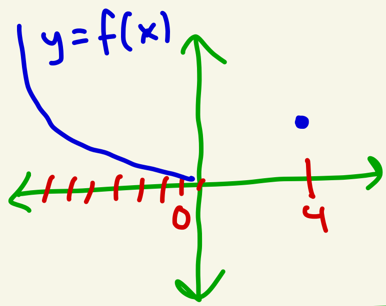
case 1: Suppose that a is a limit point of D. Then we may consider $\lim_{x \to a} f(x)$.

Then by def, f is continuous at a if and only if

(1) $\lim_{x \to a} f(x)$ exists

and (2) $\lim_{x \to a} f(x) = f(a)$

In our example above this is when $-\infty < a \leq 0$



<u>case 2:</u> Suppose a is not a limit point of D.

Then there exists $\delta > 0$ where

$$(a-\delta, a+\delta) \cap A = \{a\}$$

Then, if $x \in A$ and $|x-a| < \delta$ we have that $x = a$ and so

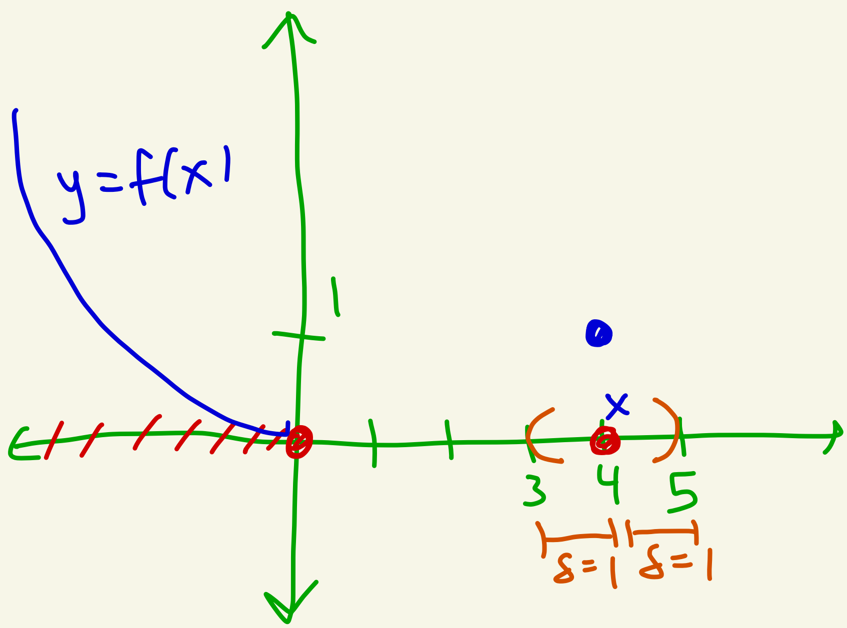
$$|f(x) - f(a)| = |f(a) - f(a)| = 0 < \varepsilon$$

for any $\varepsilon > 0$.

So, in this case f is continuous at $x = a$.

This kind of point a is called an <u>isolated point</u> of D

In our example this happens when $a = 4$.



(D is in red)

Ex: Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$.

Every $a \in \mathbb{R}$ is a limit point of $\mathbb{R}$.

Thus to show that f is continuous at $a \in \mathbb{R}$ we just need to show that $\lim_{x \to a} x^2 = a^2$.

In topic 3, we showed this for $a = 2$.

Let's do this for any $a \in \mathbb{R}$.

Claim: $\lim_{x \to a} x^2 = a^2$

pf: Let $\varepsilon > 0$.

Note that

$$|x^2 - a^2| = |x+a||x-a|$$

If $\delta \leq 1$ and $|x-a| < \delta \leq 1$, then

$$\begin{aligned} |x+a| &= |x-a+a+a| \\ &= |x-a+2a| \\ &\leq |x-a| + |2a| \\ &< 1 + 2|a| \end{aligned}$$

Let $\delta = \min\left\{\frac{\varepsilon}{1+2|a|}, 1\right\}$.

Then if $|x-a| < \delta$, we have

$$|x^2 - a^2| = |x+a|\,|x-a|$$
$$< (1+2|a|)\,|x-a|$$
$$< (1+2|a|)\frac{\varepsilon}{1+2|a|}$$
$$= \varepsilon$$

$\delta \le 1$ so $|x+a| < 1+2|a|$

$|x-a| < \delta \le \frac{\varepsilon}{1+2|a|}$

Thus, if $|x-a| < \delta$, then $|x^2 - a^2| < \varepsilon$.

So, $\lim_{x \to a} x^2 = a^2$ ∎

Ex: (Dirichlet's function – 1829)

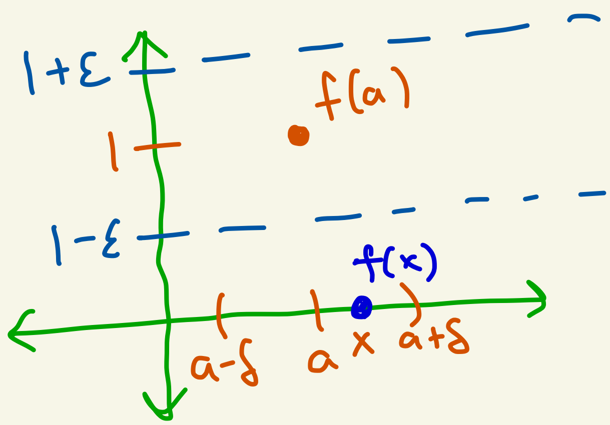
Let $f: \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Then f is not continuous at any point in $\mathbb{R}$.

Proof:

Let $a \in \mathbb{R}$ and $\varepsilon = \frac{1}{2}$.

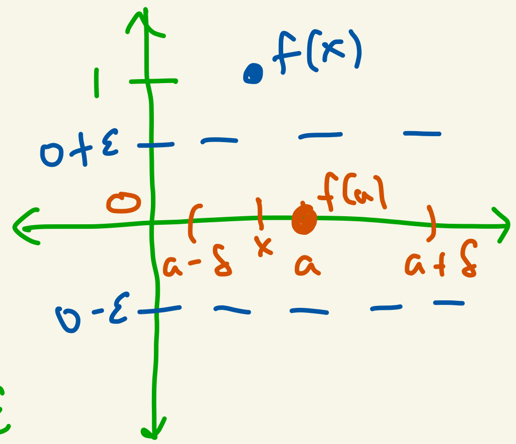


Case 1: Suppose a is rational.

Then, $f(a) = 1$.

Given any $\delta > 0$, there exists an irrational number $x \in (a-\delta, a+\delta)$ making $|f(x) - f(a)| = |0 - 1| = 1 > \varepsilon$

So f is not continuous at a.



Case 2: Suppose a is irrational.

Then $f(a) = 0$.

Given any $\delta > 0$, there exists a rational number $x \in (a-\delta, a+\delta)$ making $|f(x) - f(a)| = |1 - 0| = 1 > \varepsilon$

So f is not continuous at a. $\blacksquare$

Theorem: Let $D \subseteq \mathbb{R}$ and $a \in D$. Let $f: D \to \mathbb{R}$ and $g: D \to \mathbb{R}$ both be continuous at a. Let $\alpha \in \mathbb{R}$. Then, αf, $f+g$, $f-g$, and fg are continuous at a.

If $g(a) \neq 0$, then $\frac{f}{g}$ is continuous at a.

proof:

If a is not a limit point of D, then all the above functions are continuous at a.

Suppose a is a limit point of D. Then this theorem follows from theorems on limits.

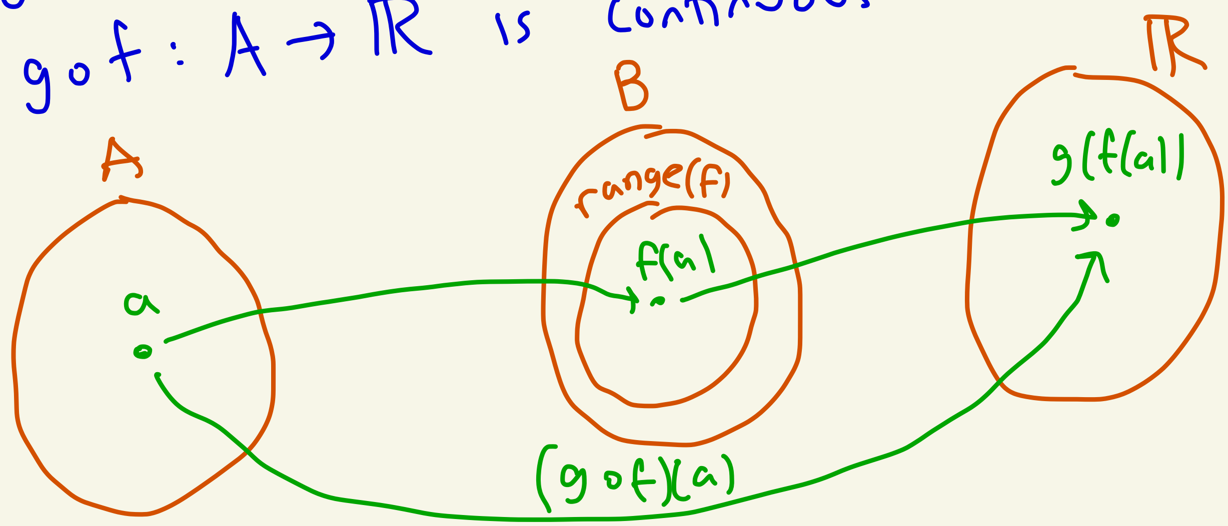
For example, since f and g are continuous at a we know $\lim_{x \to a} f(x) = f(a)$ and $\lim_{x \to a} g(x) = g(a)$.

Thus, $\lim_{x \to a} f(x)g(x) = \left[\lim_{x \to a} f(x)\right]\left[\lim_{x \to a} g(x)\right] = f(a)g(a)$

So, fg is continuous at a.

The other proofs are similar. $\boxtimes$

Theorem: Let $A, B \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ be functions such that the range of f is contained in B. If f is continuous at some point $a \in A$ and g is continuous at $f(a) \in B$, then $g \circ f : A \to \mathbb{R}$ is continuous at a.



proof:

Let $\varepsilon > 0$.

Since g is continuous at $f(a)$ there exists $\delta_1 > 0$ where if $y \in B$ and $|y - f(a)| < \delta_1$ then $|g(y) - g(f(a))| < \varepsilon$.

Since f is continuous at a there exists $\delta > 0$ where if $x \in A$ and $|x - a| < \delta$ then $|f(x) - f(a)| < \delta_1$

Since the range of f is contained in B, we have that if $x \in A$ and $|x-a|<\delta$, then $f(x) \in B$ and $|f(x)-f(a)|<\delta$, which will give $|g(f(x))-g(f(a))|<\varepsilon$.

Thus, $g \circ f$ is continuous at a. $\blacksquare$

Theorem: Let $D \subseteq \mathbb{R}$ and $f: D \to \mathbb{R}$ and $a \in D$. Then f is continuous at a if and only if $\lim_{n\to\infty} f(x_n) = f(a)$ for every sequence (x_n) contained in D with $x_n \to a$.

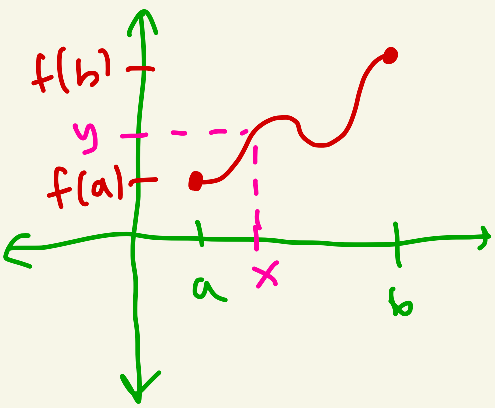
proof: HW $\blacksquare$

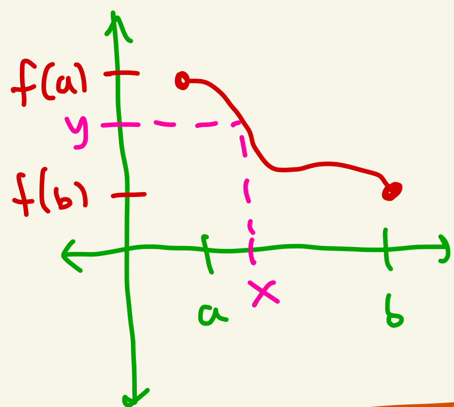
Theorem (Intermediate value theorem)

Let f be continuous on $[a,b]$ where $a<b$.
If for some $y \in \mathbb{R}$ we have either

$$f(a) < y < f(b) \quad \text{or} \quad f(b) < y < f(a)$$

then there exists x with $a<x<b$ and $f(x)=y$.





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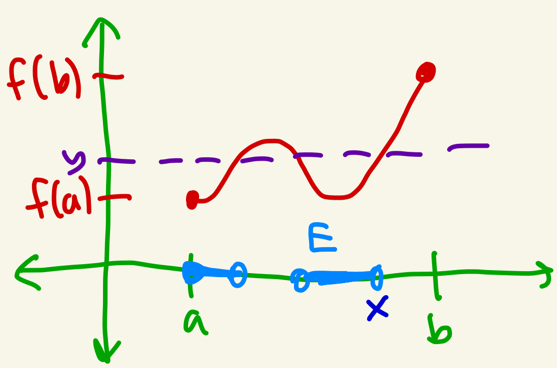
proof:

We will prove it when $f(a) < y < f(b)$. The other case is similar.

Define

$$E = \{ t \mid a \le t \le b \text{ and } f(t) < y \}$$



Note that $a \in E$ since $a \le a \le b$ and $f(a) < y$.
So, E is not empty.
Since E is bounded above by b we know $x = \sup(E)$ exists

From HW 1 we know that if $A \subseteq B$ then
$$\inf(B) \le \inf(A) \le \sup(A) \le \sup(B).$$

Since $E \subseteq [a,b]$ we know
$$a \le \inf(E) \le \sup(E) \le b$$

$\boxed{\text{So, } a \le x \le b.}$

Next we show that $a < x < b$.

We know that $y < f(b)$.

Since f is continuous at b, there exists $\delta > 0$ so that if
$t \in [a,b]$ and $|t-b| < \delta$,
then $|f(t) - f(b)| < \underbrace{f(b) - y}_{\text{positive since } y < f(b)}$

So, if $b - \delta < t \le b$, then
$$-(f(b)-y) < f(t) - f(b) < f(b) - y$$

That is if $b-\delta < t \le b$, then
$$y < f(t) < 2f(b) - y$$

So, if $b - \delta < t \le b$, then $y < f(t)$.

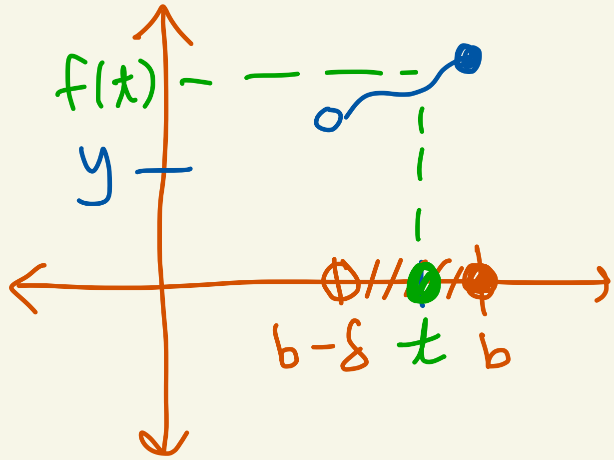
So,

$(b-\delta, b] \cap E = \phi$

Thus,

$x \leq b-\delta < b.$

So, $x < b$.



A similar computation gives $a < x$.

We know $f(a) < y$

Since f is continuous at a there exists $\delta > 0$ where if $a \leq t < a+\delta$, then

$|f(t) - f(a)| < y - f(a).$

That is, for $a \leq t < a+\delta$,

$-(y-f(a)) < f(t) - f(a) < y - f(a)$

or

$-y + 2f(a) < f(t) < y$

So, $[a, a+\delta) \subseteq E$,

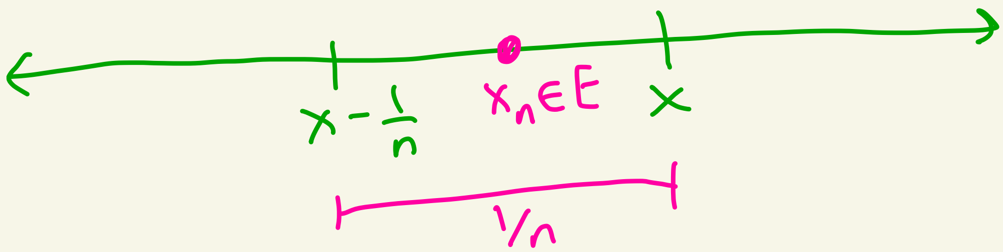
Thus since $x = \sup(E)$ we must have $a < x$.

Thus, $\boxed{a < x < b.}$

Next we will show that $f(x) = y$.

First we show that $f(x) \leq y$.

By the inf-sup theorem, since $x = \sup(E)$, for each $n \in \mathbb{N}$ there exists $x_n \in E$ such that $x - \frac{1}{n} < x_n \leq x$.



This gives us a sequence (x_n) contained in E that converges to x.

For every n, since $x_n \in E$ we know $f(x_n) < y$.

Since f is continuous at x, and x is a limit point of $E \subseteq [a,b]$, we know
$$\lim_{n \to \infty} f(x_n) = f(x).$$

By HW 2, since $f(x_n) < y$ for all n and $\lim_{n \to \infty} f(x_n) = f(x)$ we know that $f(x) = \lim_{n \to \infty} f(x_n) \leq y$.

$\boxed{\text{So, } f(x) \leq y.}$

Now we rule out the case $f(x) < y$.

Suppose $f(x) < y$,

Then since f is continuous at x

there exists $\delta > 0$ where if

$t \in [a,b]$ and $|t-x| < \delta$, then

$$|f(t) - f(x)| < \underbrace{y - f(x)}_{\substack{\text{positive} \\ \text{since } f(x)<y}}$$

In the above we can assume that $\delta < b - x$ by shrinking it if needed.

Thus, if $x - \delta < t < x + \delta \underset{\boxed{\delta < b-x}}{<} b$

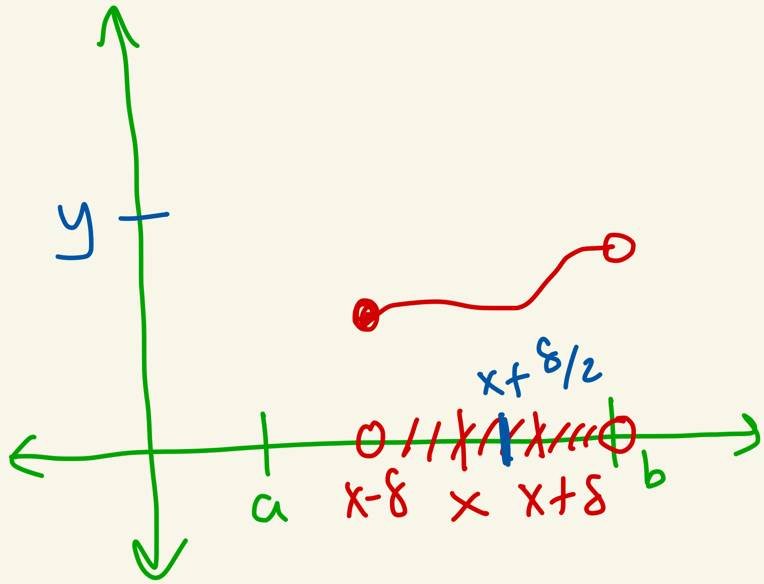
then $-(y - f(x)) < f(t) - f(x) < y - f(x)$

So if $x - \delta < t < x + \delta$,

then $-y + 2f(x) < f(t) < y$

But then

$(x-\delta, x+\delta) \subseteq E$



So for example

$x+\frac{\delta}{2} \in E$

This contradicts the fact that $x = \sup(E)$.

Hence $f(x) < y$ is impossible

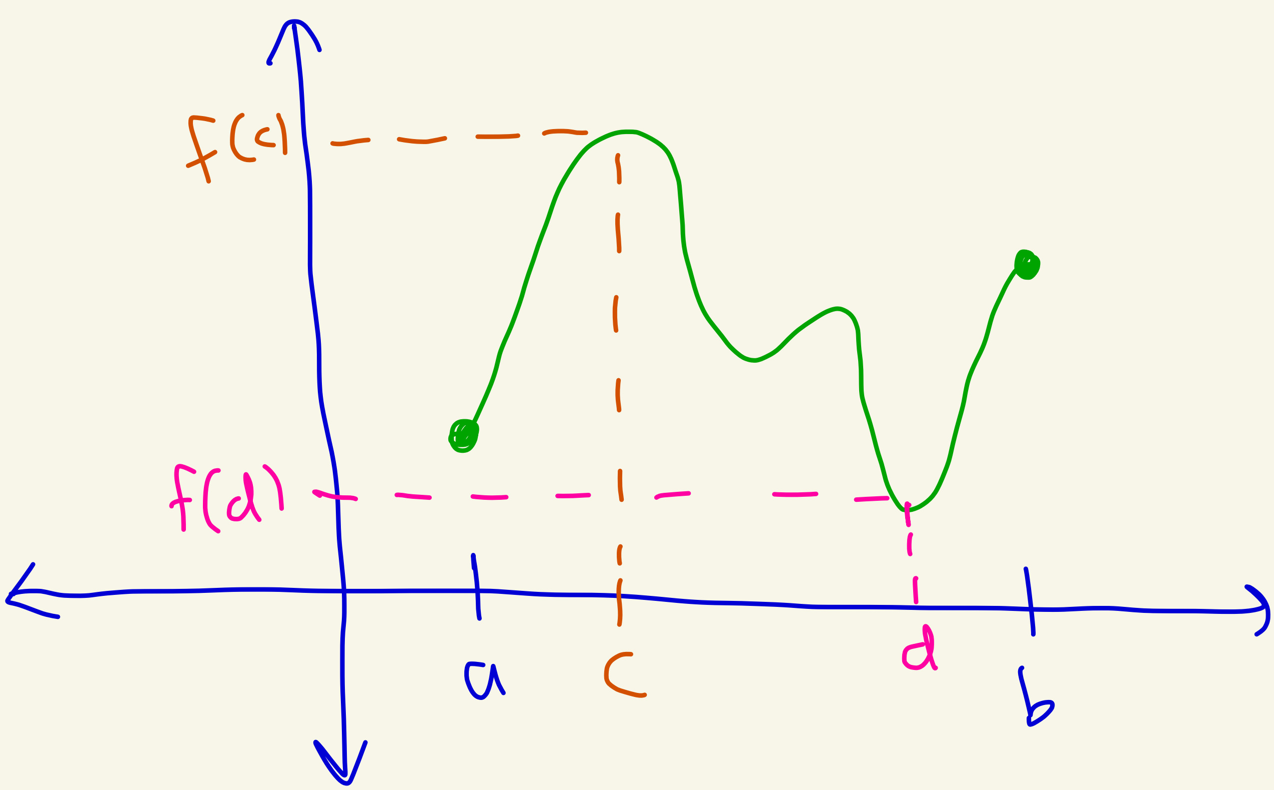
Therefore $f(x) = y$.

$\blacksquare$

Theorem: Let f be continuous on $[a,b]$ with $a<b$.
Then f attains it's maximum and minimum values on $[a,b]$.

That is, there exists $c \in [a,b]$ where $f(c) \geq f(x)$ for all $a \leq x \leq b$

And there exists $d \in [a,b]$ where $f(d) \leq f(x)$ for all $a \leq x \leq b$



proof:

We will prove that f attains its maximum. For the minimum, repeat the proof with $-f$ in place of f.

Consider

$$S = \{ f(x) \mid a \leq x \leq b \}$$

Let's show that S is bounded from above.

Suppose that it is not.

Then for each $n \in \mathbb{N}$ there exists $x_n \in [a,b]$ with $f(x_n) > n$.

Since $a \leq x_n \leq b$ for each n we get that (x_n) is a bounded sequence.

By Bolzano–Weierstrass there is a convergent subsequence

$$x_{n_1}, x_{n_2}, x_{n_3}, x_{n_4}, \ldots$$

with $n_1 < n_2 < n_3 < n_4 < \cdots$

Suppose $\lim_{n_k \to \infty} x_{n_k} = c$.

Since $a \leq x_{n_k} \leq b$ for all n_k, by HW 2 we get $a \leq c \leq b$.

Since f is continuous at c we get
$$f(c) = \lim_{x \to c} f(x)$$
$$= \lim_{n_k \to \infty} f(x_{n_k})$$

HW 4
its essentially the function-sequence limit theorem

But this is a contradiction because $f(x_{n_k}) \geq n_k$ for all k

and $n_k \to \infty$.

So, $\lim_{n_k \to \infty} f(x_{n_k})$ does not exist.

Therefore we must have that S is bounded from above.

So, $M = \sup(S)$ exists.

By the inf-sup theorem, for each $m \in \mathbb{N}$ there exists y_n with $a \leq y_m \leq b$ and

$$M - \frac{1}{m} < f(y_m) \leq M.$$

Thus, $\lim_{m \to \infty} f(y_m) = M$.

Let $\varepsilon > 0$. Pick N s.t. $\frac{1}{N} < \varepsilon$. Then if $m \geq N$, then $|f(y_m) - M| < \frac{1}{m} \leq \frac{1}{N} < \varepsilon$

Since (y_m) is a bounded sequence there exists a convergent subsequence (y_{m_ℓ}).

Suppose $y_{m_\ell} \to d$.

Since $a \leq y_{m_\ell} \leq b$ for all m_ℓ we know by HW 2 that $a \leq d \leq b$.

Thus, by the continuity of f

$$f(d) = \lim_{m_\ell \to \infty} f(y_{m_\ell}) = M.$$

HW 4

HW 2
since $f(y_m) \to M$
we know $f(y_{m_\ell}) \to M$

Since $f(d) = M = \sup(S)$ we know
$f(d) \geq f(x)$ for all $a \leq x \leq b$.

So f attains its maximum
on $[a, b]$ at d. $\blacksquare$